Rapid stabilization of general linear PDE with F-equivalence

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Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space. We consider

$$\partial_t u = \mathcal{A}u + Bw, \tag{1}$$

where

- $\mathcal{A}$ and B are linear operators;
- $u(t, \cdot) \in X$ the state of the system;
- $w(t) \in \mathbb{C}$ the control input

Feedback stabilization

Conditions on $\mathcal{A}$ and B to:

1. Exhibit a feedback law $K \in \mathcal{L}(D(\mathcal{A}), \mathbb{C})$ s.t $w(t) = Ku(t, \cdot)$ and

$$\partial_t u = \mathcal{A}u + BKu, \tag{2}$$

   is well-posed in X, i.e $\mathcal{A} + BK$ generates a C^0 semi-group in X;

2. show that any solution of (2) converges exponentially quick to 0 in X, i.e

$$\|u(t,\cdot)\|_X \le ke^{-\mu t}\|u(0,\cdot)\|_X, \quad \forall t \ge 0.$$

   equivalently, the semi-group generated by $\mathcal{A} + BK$ is exponentially stable in X.

F-equivalence method

Let $\lambda > 0$. Assume that $\mathcal{A}$ generates a **finite growth C^0 semi-group** and consider

$$\partial_t v = (\mathcal{A} - \lambda I)v. \tag{3}$$

There exists $\omega \in \mathbb{R}$, $k \geq 1$ such that

$$\|v(t,\cdot)\|_X \leq k e^{-(\lambda-\omega)t} \|v(0,\cdot)\|_X, \quad \forall t \geq 0.$$

If there exist an isomorphism T and feedback K s.t

$$\partial_t u = (\mathcal{A} + BK)u \quad \underset{v=Tu}{\Longleftrightarrow} \quad \partial_t v = (\mathcal{A} - \lambda I)v$$

then

$$\|u(t,\cdot)\|_X \leq K e^{-(\lambda-\omega)t} \|u(0,\cdot)\|_X, \quad \forall t \geq 0. \tag{4}$$

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Fredholm transformation

- KdV equation: Coron and Lü 2014 and K–S equation: Coron and Lü 2015
- Boundary-controlled systems: Coron, Hu, and Olive 2017, Deutscher and Gabriel 2019, Redaud, Auriol, and Niculescu 2022
- The linear Schrodinger equation: Coron, Gagnon, and Morancey 2018
- Degenerate parabolic: Gagnon, Lissy, and Marx 2021 and Lissy and Moreno 2023
- The transport equation: Zhang 2022
- The linearized Saint-Venant system: Coron, Hayat, et al. 2022
- The 1D heat equation: Gagnon, Hayat, et al. 2022
- The skew-adjoint system: Gagnon, Hayat, et al. 2025

Question?

Conjecture 1

Assume that the pair $(\mathcal{A}, B)$ is ***exactly controllable and admissible*** *in a certain Hilbert space H. Then, for any $\lambda > 0$ there exists a unique isomorphism-feedback pair (T, K) such that T transforms the system*

$$\partial_t u = \mathcal{A}u + BKu$$

into the system

$$\partial_t u = \mathcal{A}u - \lambda u.$$

Answer in finite dimension

Coron 2015

Theorem 1

Consider a linear finite dimension system defined as

$$\dot{y} = Ay + Bz, \tag{5}$$

where $y \in \mathbb{R}^n$, $z \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$. *Assume that the pair* (A,B) *is controllable, i.e*

$$\text{rank}\{B, A^2B, \cdots, A^{n-1}B\} = n. \tag{6}$$

Then, for any $\lambda > 0$, *there exists a unique isomorphism-feedback pair* $(T,K) \in GL(n,\mathbb{R}) \times \mathbb{R}^{1\times n}$ *such that* $TB = B$ *and* T *transforms the system*

$$\dot{y} = Ay + BKy,$$

into the exponentially stable system

$$\dot{y} = Ay - \lambda y.$$

Answer in infinite dimension?

Existing results

- **When $\mathcal{A}$ is self-adjoint:** Ludovick Gagnon, Amaury Hayat, et al. (2022). "Fredholm transformation on Laplacian and rapid stabilization for the heat equation". In: *Journal of Functional Analysis*
- **When $\mathcal{A}$ is a skew-adjoint:** Ludovick Gagnon, Amaury Hayat, et al. (2025). "Fredholm backstepping for critical operators and application to rapid stabilization for the linearized water waves". In: *Annales de l'Institut Fourier*

Can we extend these results to any general "**spectral**" operator $\mathcal{A}$?

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- **When $\mathcal{A}$ is self-adjoint:** Ludovick Gagnon, Amaury Hayat, et al. (2022). "Fredholm transformation on Laplacian and rapid stabilization for the heat equation". In: *Journal of Functional Analysis*
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Can we extend these results to any general "**spectral**" operator $\mathcal{A}$?

Spectral assumptions on $\mathcal{A}$

Assume that

1. $\mathcal{A}$ generates a dissipative C^0 semi-group on a Hilbert space X.
2. The eigenvectors φ_n of $\mathcal{A}$ form a Riesz basis of X.
3. The eigenvalues λ_n have finite multiplicity and there exists $\alpha > 1$ such that

$$n^\alpha \lesssim |\lambda_n| + 1 \lesssim n^\alpha, \quad \forall n \in \mathbb{N}^*, \tag{7}$$

$$|\lambda_n - \lambda_p| \gtrsim n^{\alpha-1}|n-p|, \quad \forall n,p \in \mathbb{N}^*. \tag{8}$$

Main result

Theorem 2

Let $\gamma \in [0, (\alpha - 1)/2)$. *If* B *satisfies*

$$c_1 \leq |\langle B, \widetilde{\varphi_n} \rangle| \leq c_2 n^\gamma, \quad \forall n \in \mathbb{N}^*, \tag{9}$$

Then, there exist a feedback K *and an isomorphism* T *such that* T *maps the system*

$$\partial_t u = \mathcal{A}u + BKu \tag{10}$$

to the system

$$\partial_t v = \mathcal{A}v - \lambda v. \tag{11}$$

For any $\mu > 0$, (10) *is exponentially stable with decay rate* $\mu > 0$.

$(\widetilde{\varphi_n})_n$ is the bi-orthogonal family for $(\varphi_n)_n$ as $\langle \varphi_n, \widetilde{\varphi_m} \rangle = \delta_{nm}$.

Extended result

Theorem 3 (Rapid stabilization in X)

Let $\gamma \in [0, (\alpha - 1)/2)$, $r \in (1/2 - \alpha + \gamma, \alpha - 1/2 - \gamma)$, *and* $B \in \mathcal{H}^{-\alpha}$ *such that*

$$c_1 n^r \leq |\langle B, \widetilde{\varphi}_n \rangle| \leq c_2 n^{r+\gamma}, \tag{12}$$

then for any $\mu > 0$, *there exists a (constructive) bounded linear feedback* K *such that the system* (2) *is exponentially stable in the study space* X *with decay rate* μ.

Relaxed controllability and admissibility

Assume that $\mathcal{A}$ is **skew-adjoint**. From Weiss and C.-Z. Xu 2011, Russell and Weiss 1994

Lemma 4

*If B is **admissible** and if the system is in addition **exactly controllable in** X, then*

$$c \leq |\langle B, \widetilde{\varphi_n}\rangle| \leq C, \; \forall\, n \in \mathbb{N}^*,$$

1. Admissibility and exact controllability in X imply (12) for $r = \gamma = 0$.
2. What if there is no exact controllability in X, i.e (12) with $r \in (1/2 - \alpha + \gamma, -\gamma)$?
   - Assumpion (12) $\quad \Rightarrow \quad |\langle B, \widetilde{\varphi_n}\rangle| \leq C n^{r+\gamma}$.
   - Exact controllability in X $\quad \Rightarrow \quad c \leq |\langle B, \widetilde{\varphi_n}\rangle|$.
   - $c \leq C n^{r+\gamma}$ is a contradiction since $r + \gamma < 0$.

Relaxed controllability and admissibility

What if there is no exact controllability in X?

The operator $\mathcal{A}$ is still assumed to be skew adjoint.

1. From Zabczyk 2020 and Trélat, Wang, and Y. Xu 2019, stabilization is not possible with $K \in \mathcal{L}(X, \mathbb{C})$.
2. From Theorem 3, stabilization is possible with feedback $K \in \mathcal{L}(D(\mathcal{A}), \mathbb{C})$.

Liu et al. 2022 and Ma, Wang, and Yu 2023 has already used $K \in \mathcal{L}(D(\mathcal{A}), \mathbb{C})$.

Conclusion

1. We provide more relaxed conditions to stabilize a general PDE
2. We weakened controllability and admissibility requirement for skew adjoint systems

Thank you for your attention!